On limit theorems for continued fractions

Zbigniew S. Szewczak *

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Abstract

It is shown that for sums of functionals of digits in continued fraction expansion the Kolmogorov-Feller weak laws of large numbers and the Khinchine-Lévy-Feller-Raikov characterization of the domain of attraction of the normal law hold.

Key words: weak law of large numbers, domain of attraction of the normal law, ψ–mixing, continued fractions.

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1 Introduction and result

Let $a_n(x)$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, denote the partial quotients (or digits) in the simple non-terminating continued fraction expansion of an irrational number $x \in (0, 1]$, (Cf. [20])

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x)+\frac{1}{a_3(x)+\dots}}} = \frac{1}{a_1(x)+}\,\frac{1}{a_2(x)+}\,\frac{1}{a_3(x)+}\cdots .$$

Let $\mathcal{B}$ denote Borel subsets of $(0, 1]$ and P denote the Gauss' measure

$$\mathrm{P}(A) = \frac{1}{\ln 2}\int_A \frac{dx}{1+x}, \qquad A \in \mathcal{B}.$$

It is well-known that the random sequence $\{a_n\}$ defined on the probability space $((0, 1], \mathcal{B}, \mathrm{P})$ is strictly stationary.

The literature concerning the limit theory for functionals of digits of continued fraction expansion (see e.g. [20], [14], [16], [17], [26], [2]) reveals that some classical results for sums of i.i.d. random variables (see e.g. [9], [25], [14], [30]) found their full analogies in this theory. One example is the Marcinkiewicz-Zygmund law of large numbers (Cf. [23], [27]), or more generally the complete convergence (Cf. [33], [26]).

In this note we discover further analogies. The first one is motivated by Lemma in [7] (see also Theorem 4.13 in [30]) and §4, Ch. VI in [23] (see also Satz XII in [22] & [21], Theorem 1 in §7, Ch. VII in [8], Theorem 1.3 in [10]).

*Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, ul. Chopina 12/18, 87-100 Toruń, Poland, e-mail: zssz@mat.uni.torun.pl

Theorem 1 *Let $c_n \to \infty$ be a sequence of positive numbers and f be a Borel function. In order that there exist a sequence $\{b_n\}$ such that*

$$c_n^{-1}(\sum_{k=1}^{n} f(a_k) - b_n) \to_{\mathrm{P}} 0 \tag{1.1}$$

it is necessary and sufficient that simultaneously

$$n\mathrm{P}[|f(a_1)| > c_n] \to 0, \qquad \frac{n}{c_n^2}\mathrm{E}[f^2(a_1) I_{[|f(a_1)| \le c_n]}] \to 0. \tag{1.2}$$

If the latter conditions are satisfied, we can set $b_n = n\mathrm{E}[f(a_1) I_{[|f(a_1)| \le c_n]}]$.

Note that by the well-known formula

$$x^2 \mathrm{P}[|Z| > x] + \mathrm{E}[|Z|^2 I_{[|Z| \le x]}] = 2\int_0^x y\mathrm{P}[|Z| > y]dy, \tag{1.3}$$

(1.2) is equivalent to

$$\int_0^1 n\mathrm{P}[|f(a_1)| > \sqrt{x} c_n]dx \to 0. \tag{1.4}$$

If there exists some additional knowledge on c_n then (1.2) can be weakened.

Theorem 2 *Suppose that c_n is a sequence of positive numbers such that*

$$\underline{\lim}_n \frac{n}{c_n^2}(c_{n+1}^2 - c_n^2) > 1. \tag{1.5}$$

Then for any Borel function f

$$(c_n^{-1}\sum_{k=1}^{n}(f(a_k) - \mathrm{E}[f(a_k) I_{[|f(a_k)| \le c_n]}]) \to_{\mathrm{P}} 0) \Longleftrightarrow (n\mathrm{P}[|f(a_1)| > c_n] \to_n 0). \tag{1.6}$$

Theorem 2 and Theorem 1.9.8 in [3] yield

Corollary 1 *Suppose that f is a Borel function, $c_n = n^{\frac{1}{r}} h(n)$, where h is a slowly varying function in the sense of Karamata and $r \in (0, 2)$. Then the relation (1.6) holds.*

Let c_n denote the accumulated entrance fees up to the n-th trial in the St. Petersburg game, i.e. $c_n = n \log_2 n$ (Cf. [7]). Since $n \ln 2\mathrm{P}[a_1 > n] \sim 1$, therefore $\mathrm{E}[a_1 I_{[a_1 \le c_n]}] \sim \log_2 n$ and by Corollary 1 (see also Theorem 3 in [35]) we get for $f(x) = x$

$$c_n^{-1}\sum_{k=1}^{n} a_k \to_{\mathrm{P}} 1.$$

On the other hand, by Theorem 4 in [35], the convergence in probability cannot be replaced by the almost sure one.

The second analogy is motivated by the famous characterization of the domain of attraction of the normal law due to Khinchine (Cf. [19]), Lévy (Cf. [24]) and Feller (Cf. [6]) (see also Corollary 1, §5, Ch. XVII in [8])

Theorem 3 *Let f be a Borel function. In order that there exist sequences $\{c_n\}$ and $\{d_n\}$ such that* $\lim_{n\to\infty} \mathcal{L}(c_n^{-1}(\sum_{k=1}^n f(a_k) - d_n)) = \mathcal{N}(0,1)$ *it is necessary and sufficient that the function* $\mathrm{E}[f^2(a_1) I_{[|f(a_1)|\le x]}]$ *is slowly varying in the sense of Karamata. If the latter condition is satisfied, we can set* $d_n = n\mathrm{E}[f(a_1)]$.

Theorem 3 improves Proposition 2.9 (II) in [32] obtained under additional assumptions on c_n and d_n. Furthermore, we no longer have to assume that condition $(*)$ on page 56 of [32] is satisfied. By Theorem 3 and the results in [35] we get that functionals of digits in continued fraction expansion satisfy the Raikov principle (Cf. Twierdzenie 4, §28, Ch. V in [9]), namely

Corollary 2 *Suppose that f is a Borel function and c_n is a positive sequence. Then* $c_n^{-2}\sum_{k=1}^n f^2(a_k) \to_{\mathrm{P}} 1$ *and* $\mathrm{E}[f(a_1)] = 0$ *if and only if*

$$\lim_{n\to\infty} \mathcal{L}(c_n^{-1}\sum_{k=1}^n f(a_k)) = \mathcal{N}(0,1).$$

As it can be concluded from the rest of this note, the presented results remain true in a little bit more general mathematical environment.

2 Preliminaries

We group here different results that will be used later on. Let $\{X_k\}_{k\in\mathbb{N}}$ be a random sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, denote by $\{\tilde{X}_k\}$ it's independent copy and in the case of stationarity by $\{X_k^*\}$ it's i.i.d. associated sequence (all sequences are sharing the same probability space). Define

$$S_n = \sum_{k=1}^n X_k, \quad \widehat{X}_k = X_k - \tilde{X}_k, \quad \widehat{S}_n = \sum_{k=1}^n \widehat{X}_k,$$

$$M_n = \max_{1\le k\le n} |X_k|, \quad \widehat{M}_n = \max_{1\le k\le n} |\widehat{X}_k|, \quad M_n^* = \max_{1\le k\le n} |X_k^*|.$$

Denote by $\mathcal{F}_k^m$ the σ–field generated $X_k, X_{k+1}, \ldots, X_m$, $m \in \mathbb{N}$, and recall the following coefficients of dependence

$$\psi_n = \sup_{k\in\mathbb{N}} \sup\{|\frac{P(A\cap B)}{P(A)P(B)} - 1|;\ P(A)P(B) > 0,\ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{n+k}^\infty\};$$

$$\psi_n^* = \sup_{k\in\mathbb{N}} \sup\{\frac{P(A\cap B)}{P(A)P(B)};\ P(A)P(B) > 0,\ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{n+k}^\infty\};$$

$$\psi_n' = \inf_{k\in\mathbb{N}} \inf\{\frac{P(A\cap B)}{P(A)P(B)};\ P(A)P(B) > 0,\ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{n+k}^\infty\};$$

$$\varphi_n = \sup_{k\in\mathbb{N}} \sup\{|P(B\,|\,A) - P(B)|;\ P(A) > 0,\ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{n+k}^\infty\};$$

$$\rho_n = \sup_{k\in\mathbb{N}} \sup\{|Corr(f,g)|;\ f \in \mathrm{L}^2_{\mathrm{real}}(\mathcal{F}_1^k), g \in \mathrm{L}^2_{\mathrm{real}}(\mathcal{F}_{n+k}^\infty)\}.$$

It is well-known that

$$\psi_n = \max\{\psi_n^* - 1, 1 - \psi_n'\}, \psi_n \geq 2\varphi_n, \psi_n \geq \rho_n, 1 - \varphi_n \geq \psi_n', 4\varphi_n \geq (\rho_n)^2, \quad (2.7)$$

for every $n \geq 1$ (Cf. [5], p.109). By Theorem 5.2 in [5] the symmetrized $\psi_n, \psi_n^*, \psi_n', \varphi_n, \rho_n$ coefficients of the sequence $\{\widehat{X}_k\}$, say $\widehat{\psi}_n, \widehat{\psi}_n^*, \widehat{\psi}_n', \widehat{\varphi}_n, \widehat{\rho}_n$, satisfy

$$\widehat{\psi}_n \leq (1+\psi_n)^2 - 1, \widehat{\psi}_n^* \leq (\psi_n^*)^2, \widehat{\psi}_n' \geq (\psi_n')^2, \widehat{\varphi}_n \leq 1 - (1-\varphi_n)^2, \widehat{\rho}_n \leq \rho_n, \quad (2.8)$$

for every $n \geq 1$.

The following lemma is a dependent version of Lévy's inequality.

Lemma 1 *Assume that $\mathcal{L}(S_n - S_k)$ are symmetric for $n > k \geq 1$. Then for $n \geq 1$ and $x > 0$*

$$2P[|S_n| > x] \geq \psi_1' P[\max_{1 \leq k \leq n} |S_k| > x].$$

Proof of Lemma 1

Consider the sets

$$C_1^+ = [S_1 > x], \quad C_k^+ = [|S_1| \leq x, |S_2| \leq x, \ldots, |S_{k-1}| \leq x, S_k > x],$$

$$C_1^- = [-S_1 > x], \quad C_k^- = [|S_1| \leq x, |S_2| \leq x, \ldots, |S_{k-1}| \leq x, -S_k > x],$$

and set $C_k = C_k^+ \cup C_k^-$, $C = \bigcup_{k=1}^n C_k$. Since the C_k are disjoint we obtain

$$\begin{aligned}
2P[|S_n| > x] &= 2P[\max_{1 \leq k \leq n} |S_k| > x, |S_n| > x] \\
&= 2\sum_{k=1}^n P(C_k \cap [|S_n| > x]) \\
&\geq 2\sum_{k=1}^n (P(C_k^+ \cap [S_n > x]) + P(C_k^- \cap [-S_n > x])) \\
&\geq 2\sum_{k=1}^n (P(C_k^+ \cap [S_n - S_k \geq 0]) + P(C_k^- \cap [-S_n + S_k \geq 0])) \\
&\geq \psi_1' \sum_{k=1}^n (P(C_k^+) + P(C_k^-)) = \psi_1' \sum_{k=1}^n P(C_k) \\
&= \psi_1' P(C) = \psi_1' P[\max_{1 \leq k \leq n} |S_k| > x],
\end{aligned}$$

because $2P[S_n - S_k \geq 0] = 2P[S_n - S_k \leq 0] = 1 + P[S_n - S_k = 0] \geq 1$. This proves Lemma 1. $\square$

It is well-known that if $E|X| < \infty$, $E|Y| < \infty$, and X is $\mathcal{F}_1^k$ measurable while Y is $\mathcal{F}_{n+k}^\infty$ measurable then

$$|E[XY] - E[X]E[Y]| \leq \psi_n E|X|E|Y| \quad (2.9)$$

(Cf. Lemma 1.2.11 in [26]).

The statement below is a dependent version of Kolmogorov's inverse inequality (Cf. [25], p.235, [1], Theorem 2.8).

Lemma 2 *Let $\{X_k\}$ be a strictly stationary sequence such that $|X_k| \le c$ and $\mathcal{L}(S_k)$ are symmetric. Then for every $x > 0$*

$$P[\max_{1\le k\le n} |S_k| > x] \ge \frac{\psi_1'(E[S_n^2] - x^2)}{4(1+\psi_1)E[S_n^2] + \psi_1'(2(c+x)^2 - x^2)}, \qquad n \ge 1.$$

Proof of Lemma 2

Let C_k and C be as in the proof of Lemma 1. We have

$$E[S_n^2 I_{C_k}] \le 2E[(S_n - S_k)^2 I_{C_k}] + 2E[S_k^2 I_{C_k}].$$

By (2.9) we get

$$E[(S_n - S_k)^2 I_{C_k}] \le (1+\psi_1)E[(S_n - S_k)^2]P(C_k).$$

Since $|S_k I_{C_k}| \le (c+x)I_{C_k}$ so by stationarity and Lemma 1 we obtain ($S_0 = 0$)

$$\begin{aligned}
&E[S_n^2 I_C] \\
&\quad \le 2\sum_{k=1}^{n}((1+\psi_1)E[(S_n-S_k)^2]P(C_k) + (c+x)^2 P(C_k)) \\
&\quad \le 2\sum_{k=1}^{n} P(C_k)((1+\psi_1)\max_{1\le k\le n} E[(S_n - S_k)^2] + (c+x)^2) \\
&\quad = 2\sum_{k=1}^{n} P(C_k)((1+\psi_1)\max_{1\le k< n} E[S_k^2] + (c+x)^2) \\
&\quad \le 2\sum_{k=1}^{n} P(C_k)((1+\psi_1)E[\max_{1\le k\le n} S_k^2] + (c+x)^2) \\
&\quad \le 2\sum_{k=1}^{n} P(C_k)(\frac{2(1+\psi_1)}{\psi_1'}E[S_n^2] + (c+x)^2) \\
&\quad = 2P(C)(\frac{2(1+\psi_1)}{\psi_1'}E[S_n^2] + (c+x)^2).
\end{aligned}$$

On the other hand

$$E[S_n^2 I_C] = E[S_n^2] - E[S_n^2 I_{\Omega\setminus C}] \ge E[S_n^2] + x^2 P(C) - x^2.$$

This completes the proof. □

The next inequality follows from the proof on p.298 in [29].

Proposition 1 *Suppose $\{X_k\}$ is a strictly stationary sequence and $\varphi_m < 1$. Then for every $x \ge 0$ and every $n \ge m \ge 1$*

$$(1-\varphi_m)P[M^*_{\lfloor n/m\rfloor} > x] \le P[M_n > x] \le m(1+\varphi_m)P[M^*_{\lfloor n/m\rfloor+1} > x].$$

We will need the following estimate

Lemma 3 *Suppose $\{X_k\}$ is a strictly stationary sequence and $\psi_1' > 0$. Then for every $x \ge 0$ and every $n \ge m \ge 1$*

$$(1-\widehat{\varphi}_1)\widehat{\psi}_1'(1 - e^{-nP[\widehat{X}_1 > 2x]}) \le 4mP[|\widehat{S}_{\lfloor n/m\rfloor+1}| > x].$$

PROOF OF LEMMA 3

It is easy to see that for every $x \geq 0$ we have

$$P[M_n > 2x] \leq P[\max_{1\leq k\leq n} |S_k| > x]. \tag{2.10}$$

Therefore by Proposition 1 and Lemma 1 we obtain

$$\begin{aligned}(1-\widehat{\varphi}_1)\widehat{\psi}_1'(1-e^{-nP[\widehat{X}_1>2x]}) &\leq (1-\widehat{\varphi}_1)\widehat{\psi}_1' P[\max_{1\leq k\leq n} |\widehat{X}_k^*| > 2x] \\ &\leq \widehat{\psi}_1' P[\widehat{M}_n > 2x] \leq 2m\widehat{\psi}_1' P[\widehat{M}_{\lfloor n/m\rfloor+1} > 2x] \\ &\leq 2m\widehat{\psi}_1' P[\max_{1\leq k\leq \lfloor n/m\rfloor+1} |\widehat{S}_k| > x] \leq 4mP[|\widehat{S}_{\lfloor n/m\rfloor+1}| > x].\end{aligned}$$

This is our assertion. □

The following estimates are consequences of Lemma 2.1 and 2.3 in [4].

Proposition 2 *Suppose $\rho_1 < 1$ and $\sum_{n=1}^{\infty} \rho_{2^n} < \infty$ for a strictly stationary sequence $\{X_k\}$ with finite variances. Then there exist positive constants C, D depending only on $\{\rho_k\}$ such that $CnVar[X_1] \leq Var[S_n] \leq DnVar[X_1]$, $n \geq 1$.*

The proof of the next statement is an easy consequence of the hint on p.91 in [1] and it is included here for the reader convenience.

Lemma 4 *Suppose that $\{X_k\}$ is a strictly stationary sequence such that $\psi_1' > 0$ and $\mathcal{L}(c_n^{-1}(S_n - b_n))$ are asymptotically normal for $c_n^2 = nh(n)$ where $h(n)$ is a slowly varying sequence. Then $\sup_n c_n^{-q} E|\widehat{S}_n|^q < \infty$, for any $q \in (0, 2)$.*

PROOF OF LEMMA 4

Suppose $d > 0$ and $\delta \in (0, \frac{1}{2}\widehat{\psi}_1'(1-\widehat{\varphi}_1))$ are such that

$$P[c_n^{-1}|\widehat{S}_n| > d] \leq \delta,$$

for $n > N_\delta$. By Lemma 1 we have for $n > N_\delta$

$$P[c_{mn}^{-1} \max_{1\leq k\leq m} |\widehat{S}_{nk} - \widehat{S}_{n(k-1)}| > d] \leq \frac{2}{\widehat{\psi}_1'} P[c_{mn}^{-1}|\widehat{S}_{mn}| > d] \leq \frac{2\delta}{\widehat{\psi}_1'}.$$

On the other hand by Proposition 1

$$\begin{aligned}&P[c_{mn}^{-1} \max_{1\leq k\leq m} |\widehat{S}_{nk} - \widehat{S}_{n(k-1)}| > d] \\ &\quad\geq (1-\widehat{\varphi}_1)P[c_{mn}^{-1} \max_{1\leq k\leq m} |\widehat{S}_{n,k}^*| > d) \\ &\quad= (1-\widehat{\varphi}_1)(1-(1-P[c_{mn}^{-1}|\widehat{S}_n| > d])^m),\end{aligned}$$

where $\widehat{S}_{n,k}^*$ are independent copies of $\widehat{S}_{nk} - \widehat{S}_{n(k-1)}$. Thus

$$P[c_{mn}^{-1}|\widehat{S}_n| > d] \leq 1 - (1 - \frac{2\delta}{\widehat{\psi}_1'(1-\widehat{\varphi}_1)})^{\frac{1}{m}}.$$

From

$$1-(1-a)^{\frac{1}{m}} \sim \frac{1}{m}\ln\frac{1}{1-a}, \quad m\to\infty, \quad 0<a<1,$$

it follows that there exists a constant $A = A(\delta, \widehat{\psi}_1', \widehat{\varphi}_1, N_\delta)$ such that

$$P[c_{mn}^{-1}|\widehat{S}_n| > d] \leq \frac{A}{m}\cdot \tag{2.11}$$

Since $c_n^2 = nh(n)$, where $h(n)$ is a slowly varying sequence so by the Uniform Convergence Theorem for R (Cf. [3], Theorem 1.5.2, p.22) we have that for every $\gamma > 0$ there exists $N_\gamma > N_\delta$ such that

$$\left|\frac{m^{-\gamma}n^{-\gamma}h(mn)}{n^{-\gamma}h(n)} - m^{-\gamma}\right| = \left|\frac{c_{mn}}{m^{\frac{1}{2}+\gamma}c_n} - \frac{1}{m^\gamma}\right| < \frac{1}{2},$$

for $n \geq N_\gamma$ uniformly in $m \geq 1$. Thus

$$\frac{c_{mn}}{c_n} < m^{\frac{1}{2}+\gamma}(m^{-\gamma} + \frac{1}{2}) \leq \frac{3}{2}m^{\frac{1}{2}+\gamma},$$

for $n \geq N_\gamma$, $m \geq 1$ which with (2.11) gives

$$mP[c_n^{-1}|\widehat{S}_n| > \frac{3}{2}dm^{\frac{1}{2}+\gamma}] \leq A,$$

for $n \geq N_\gamma$, $m \geq 1$. Replacing m with $m^{\frac{2}{1+2\gamma}}$ we obtain

$$m^{\frac{2}{1+2\gamma}}P[c_n^{-1}|\widehat{S}_n| > md'] \leq A',$$

where $A' = A'(A, \gamma)$, $d' = d'(d, \gamma)$ and $n \geq N_\gamma$, $m \geq 1$. Now, taking γ such that $0 < \gamma < \frac{1}{q} - \frac{1}{2}$ we get

$$m^{-1+q}P[c_n^{-1}|\widehat{S}_n| > md'] \leq \frac{A'}{m^{1+c}},$$

where $c = \frac{2}{1+2\gamma} - q > 0$. Thus

$$\sum_{m=1}^{\infty} m^{q-1}P[c_n^{-1}|\widehat{S}_n| > md'] < \infty,$$

for $n \geq N_\gamma$. This completes the proof. □

The next inequality follows from Proposition 1 and Proposition 4.3, Lemma 4.2 in [12].

Proposition 3 *Suppose that $\{X_k\}$ is a strictly stationary random sequence such that $\phi_1 < 1$ and $\sup\{x; P[|X_1| \leq x] < 1\} = \infty$. Then for every $q, x > 0$*

$$(1 - \phi_1)\frac{nE[|X_1|^q I_{[|X_1|>x]}]}{1 + nP[|X_1| > x]} \leq E[M_n^q], \qquad n \geq 1.$$

The lemma below is a dependent analog of Khinchine's inequality (Cf. [1], p.176).

Lemma 5 *Suppose that $\{X_k\}$ is a random sequence such that $|X_k| < c$ and $0 < \psi_1' \le \psi_1^* < \infty$. Then for any $q \in [1, p)$ there exists a constant B_{pq} depending only on ψ_1', ψ_1^*, p, q such that*

$$E|\widehat{S}_n|^p \le B_{pq} \max\{E^{\frac{p}{q}}|\widehat{S}_n|^q, c^p\}, \qquad n \ge 1.$$

Proof of Lemma 5

From the Nagaev generalization of the inequality (3.3) in [13] (Cf. the relation (11) and the proof of Lemma in [28]) it follows that for any random sequence $\{X_k\}$ such that $|X_k| < c$ we have for $t \ge c$

$$P[\max_{1\le k\le n} |S_k| > 4t] \le \psi_1^*(P[\max_{1\le k\le n} |S_k| > t])^2. \tag{2.12}$$

In view of this and Lemma 1 we get for $t > 2c$

$$P[|\widehat{S}_n| > 4t] \le (\gamma P[|\widehat{S}_n| > t])^2,$$

where $\gamma = \frac{2\sqrt{\widehat{\psi}_1^*}}{\widehat{\psi}_1'}\cdot$ If we set

$$t_0 = 4^{\frac{1}{q}}\gamma^2 \max\{E^{\frac{1}{q}}|\widehat{S}_n|^q, c\} > 2c$$

then by the Markov inequality we obtain for $t \ge t_0$

$$\gamma^2 P[|\widehat{S}_n| > t] \le \frac{1}{4}\cdot$$

From this we have

$$\begin{aligned}
E|\widehat{S}_n|^p &= p\int_0^{4t_0} x^{p-1}P[|\widehat{S}_n| > x]dx + \\
&\quad + p\sum_{k=1}^{\infty}\int_{4^k t_0}^{4^{k+1}t_0} x^{p-1}P[|\widehat{S}_n| > x]dx \\
&\le (4t_0)^p + t_0^p\sum_{k=1}^{\infty} 4^{p(k+1)}P[|\widehat{S}_n| > 4^k t_0] \\
&\le (4t_0)^p + 4^p t_0^p \sum_{k=1}^{\infty} 4^{kp}\gamma^{\sum_{i=1}^k 2^i} P^{2^k}[|\widehat{S}_n| > t_0] \\
&\le (4t_0)^p + 4^p t_0^p\gamma^{-2}\sum_{k=1}^{\infty} 4^{kp}(\gamma^2 P[|\widehat{S}_n| > t_0])^{2^k} \\
&\le (4t_0)^p + 4^p t_0^p \gamma^{-2}\sum_{k=1}^{\infty}\frac{4^{kp}}{4^{2^k}} \\
&= (4t_0)^p(1 + \frac{1}{\gamma^2})\sum_{k=1}^{\infty}\frac{4^{kp}}{4^{2^k}} \\
&\le 4^{\frac{p}{q}} 2\gamma^{2p}4^p\sum_{k=1}^{\infty}\frac{4^{kp}}{4^{2^k}}\max\{E^{\frac{p}{q}}|\widehat{S}_n|^q, c^p\} \\
&= B_{pq}\max\{E^{\frac{p}{q}}|\widehat{S}_n|^q, c^p\}.
\end{aligned}$$

This is desired conclusion. □

We will also need the following symmetrization result for slowly varying functions.

Lemma 6 *If $E[\widehat{X}^2 I_{[|\widehat{X}|\le x]}]$ varies slowly then $E[X^2 I_{[|X|\le x]}]$ varies slowly, too.*

Proof of Lemma 6

Since $E|\widehat{X}| < \infty$ thus $E|X| < \infty$ (Cf. [25], p.243). By $P[|X| \ge 2E|X|] \le \frac{1}{2}$ we have that $median(X) \le 2E|X|$. Hence by the weak symmetrization inequalities (Cf. [25], p.245) we have

$$\frac{1}{2}P[|X| > x + 2E|X|] \le P[|\widehat{X}| > x] \le 2P[|X| > \frac{1}{2}x].$$

Therefore, if $x \ge eE|X|$ then

$$\begin{aligned}\frac{x^2 P[|X| > x]}{2\int_0^x yP[|X| > y]dy} \quad \le \quad & \frac{16x^2}{(x-2E|X|)^2}\frac{(x-2E|X|)^2 P[|\widehat{X}| > x - 2E|X|]}{2\int_0^{x-2E|X|} yP[|\widehat{X}| > y]dy} \\ & \times\frac{\int_0^{x-2E|X|} yP[|\widehat{X}| > y]dy}{\int_0^{2x} yP[|\widehat{X}| > y]dy}. \qquad (2.13)\end{aligned}$$

Since the fraction standing left of the formula number (2.13) is at most 1 thus by (1.3),(2.13) and Theorem 2, VIII, §9 in [8]

$$\frac{x^2 P[|X| > x]}{x^2 P[|X| > x] + E[X_1^2 I_{[|X|\le x]}]} = \frac{x^2 P[|X| > x]}{2\int_0^x yP[|X| > y]dy} \to 0.$$

This proves Lemma 6. □

By Lemma 2.1 in [31] and Corollary in [15] the sequence $\{f(a_k)\}$ fulfills $\psi_n \le \varrho^n$, for some $\varrho < 0.8$ and $\psi_1 \le 2\ln 2 - 1 < 0.39$. Therefore, by (2.7) and (2.8) we have $\varphi_1 < 1/2$, $\widehat{\varphi}_1 < 1/2$, $\psi_1' > 0$, $\widehat{\psi}_1' > 0$, $\psi_1^* < 2$, $\widehat{\psi}_1^* < 2$, $\widehat{\rho}_1 \le \widehat{\psi}_1 < 1$.

In particular, by Lemma 1 and (2.10) we have for every $x \ge 0$ and $n \ge 1$

$$\mathrm{P}[\max_{1\le k\le n} |\widehat{f}(a_k)| > 2x] \le \mathrm{P}[\max_{1\le k\le n} |\widehat{S}_k| > x] \le 6\mathrm{P}[|\widehat{S}_n| > x]. \qquad (2.14)$$

3 Proofs

Proof of Theorem 1

Without the loss of generality we may assume that $\mathrm{E}[f^2(a_1)] = \infty$ (the case $\mathrm{E}[f^2(a_1)] < \infty$ is covered by Theorem 3). Assume first that (1.2) holds. Set $X_{nk} = f(a_k)I_{[|f(a_k)|\le c_n]}$. By Proposition 2 we obtain

$$\mathrm{E}(\sum_{k=1}^n (X_{nk} - \mathrm{E}[X_{n1}]))^2 \le Dn\mathrm{Var}[X_{n1}].$$

In view of this and Chebyshev's inequality we get for any $\epsilon > 0$

$$\mathrm{P}[|\sum_{k=1}^n (f(a_k) - \mathrm{E}[f(a_k)I_{[|f(a_k)|\le c_n]}])| > \epsilon c_n]$$

$$\le \mathrm{P}[|\sum_{k=1}^n f(a_k)I_{[|f(a_k)|>c_n]}| > \frac{\epsilon}{2}c_n] + \mathrm{P}[|\sum_{k=1}^n (X_{nk} - \mathrm{E}[X_{nk}])| > \frac{\epsilon}{2}c_n]$$
$$\le n\mathrm{P}[|f(a_1)| > c_n] + 8Dn\epsilon^{-2}c_n^{-2}\mathrm{E}[f(a_1)^2 I_{[|f(a_1)|\le c_n]}] \to 0.$$

Conversely, assume (1.1). By (2.14)

$$\mathrm{P}[\max_{1\le k\le n} |\widehat{f}(a_k)| > \epsilon c_n] \to 0$$

so that denoting $Y_{nk}(\epsilon) = \widehat{f}(a_k)I_{[|\widehat{f}(a_k)|\le \epsilon c_n]}$ we get

$$c_n^{-1}\sum_{k=1}^n Y_{nk}(\epsilon) \to_{\mathrm{P}} 0, \qquad \epsilon \in (0,1).$$

Let $Z_n(\epsilon) = \sum_{k=1}^n Y_{nk}(\epsilon)$ then $c_n^{-1}\widehat{Z}_n(\epsilon) \to_{\mathrm{P}} 0$. Thus by Lemma 1

$$c_n^{-1} \max_{1\le k\le n} \sum_{m=1}^k \widehat{Y}_{nm}(\epsilon) \to_{\mathrm{P}} 0$$

and by Lemma 2 (with $x = \epsilon, X_k = c_n^{-1}\widehat{Y}_{nk}$) we get

$$\lim c_n^{-2}\mathrm{E}[\widehat{Z}_n^2(\epsilon)] = 0.$$

Since $\mathrm{E}[\widehat{f}^2(a_1)] = 2\mathrm{E}[f^2(a_1)] = \infty$ thus $\mathrm{E}^2|Y_{n1}(\epsilon)| = o(\mathrm{E}[Y_{n1}^2(\epsilon)])$ (Cf. (2.6.14) in [14]). Therefore by Proposition 2

$$\lim_n \frac{n}{c_n^2}\mathrm{E}[Y_{n1}^2(\epsilon)] = \lim_n \frac{n}{c_n^2}\mathrm{Var}[Y_{n1}(\epsilon)] = \lim_n \frac{n}{2c_n^2}\mathrm{Var}[\widehat{Y}_{n1}(\epsilon)] = 0, \quad \epsilon \in (0,1)$$

so that $n^{-1}c_n^2 \to \infty$. Further, by (1.3) and since $n\mathrm{P}[|\widehat{f}(a_1)| > \epsilon c_n] \to 0$ hence

$$\int_0^1 n\mathrm{P}[|f(a_1) - \tilde{f}(a_1)| > \sqrt{x\epsilon}c_n]dx \to 0, \qquad \epsilon \in (0,1).$$

Now, by the weak symmetrization inequalities (Cf. [25], p.245) we have

$$\begin{aligned}
&\int_0^1 n\mathrm{P}[|f(a_1)| > \sqrt{x}c_n]dx \\
&\quad \le \int_0^1 n\mathrm{P}[|f(a_1) - median(f(a_1))| > \sqrt{x}\frac{c_n}{2}]dx \\
&\qquad + \int_0^1 n\mathrm{P}[|median(f(a_1))| > \sqrt{x}\frac{c_n}{2}]dx \\
&\quad \le \int_0^1 2n\mathrm{P}[|f(a_1) - \tilde{f}(a_1)| > \sqrt{x}\frac{c_n}{2}]dx \\
&\qquad + \int_0^1 n\mathrm{P}[|median(f(a_1))| > \sqrt{x}\frac{c_n}{2}]dx \\
&\quad \le \int_0^1 2n\mathrm{P}[|f(a_1) - \tilde{f}(a_1)| > \sqrt{x}\frac{c_n}{2}]dx + 4\frac{n}{c_n^2}(median(f(a_1)))^2 \to 0.
\end{aligned}$$

This proves Theorem 1.

Proof of Theorem 2

By (1.5) there exists $N \in \mathbb{N}$ such that for $n > N$

$$\frac{n-1}{c_{n-1}^2}(c_n^2 - c_{n-1}^2) \geq c > 1.$$

Setting, if necessary $c_k^2 = (c+1)^{-N+k+1} c_{N-1}^2$ for $1 \leq k < N$, we may assume that

$$\frac{n-1}{c_{n-1}^2}(c_n^2 - c_{n-1}^2) \geq c, \quad n > 1, \quad c > 1, \tag{3.15}$$

or, equivalently

$$\frac{c_n^2}{n} - \frac{c_{n-1}^2}{n-1} \geq \frac{c-1}{c}\frac{c_n^2 - c_{n-1}^2}{n}, \quad n > 1, \quad c > 1. \tag{3.16}$$

Further, by (3.15) we have

$$\ln c_n^2 \geq \ln c_1^2 + \sum_{k=2}^{n}(\ln(1 + \frac{c}{k-1}) - \frac{c}{k-1}) + c\sum_{k=2}^{n}\frac{1}{k-1}$$

and since by the Taylor expansion of $\ln(1+x)$ the first sum on the right hand side is $O(1)$ so that $\underline{\lim}_n \frac{\ln \frac{c_n^2}{n}}{\ln n} \geq c - 1 > 0$. By this, (3.15) and (3.16) we get $\frac{c_n^2}{n} \nearrow \infty$.

Using in (3.16) the convention $\frac{0}{0} = 0$ we have

$$\begin{aligned}
&\frac{n}{c_n^2}\mathrm{E}[f(a_1)^2 I_{[|f(a_1)| \leq c_n]}] \\
&\quad \leq \frac{n}{c_n^2}\sum_{k=1}^{n} c_k^2 \mathrm{P}[|f(a_1)| \in (c_{k-1}, c_k]] \\
&\quad = \frac{n}{c_n^2}\sum_{k=1}^{n}\sum_{\nu=1}^{k}(c_\nu^2 - c_{\nu-1}^2)\mathrm{P}[|f(a_1)| \in (c_{k-1}, c_k]] \\
&\quad = \frac{n}{c_n^2}\sum_{\nu=1}^{n}(c_\nu^2 - c_{\nu-1}^2)\sum_{k=\nu}^{n}\mathrm{P}[|f(a_1)| \in (c_{k-1}, c_k]] \\
&\quad \leq \frac{n}{c_n^2}\sum_{\nu=1}^{n}(\frac{c_\nu^2 - c_{\nu-1}^2}{\nu})\nu\mathrm{P}[|f(a_1)| > c_{\nu-1}] \\
&\quad \leq \frac{c}{c-1}\frac{n}{c_n^2}\sum_{\nu=1}^{n}(\frac{c_\nu^2}{\nu} - \frac{c_{\nu-1}^2}{\nu-1})\nu\mathrm{P}[|f(a_1)| > c_{\nu-1}] \to_n 0
\end{aligned}$$

by the Toeplitz lemma (Cf. [25], p.238). In view of Theorem 1 the relation (1.6) is proved.

Proof of Theorem 3

Suppose that $\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\le x]}]$ is slowly varying and $\mathrm{E}[f(a_1)]=0$. Define the sequence b_n as follows: if $\mathrm{E}[f^2(a_1)]<\infty$ then $b_n=\sigma\sqrt{n}$ provided that

$$\infty>\sigma^2=\mathrm{E}[f^2(a_1)]+2\sum_{k=2}^{\infty}\mathrm{E}[f(a_1)f(a_k)]>0;$$

if $\mathrm{E}[f^2(a_1)]=\infty$ then

$$b_n=\sup\{x>0\,;\,x^{-2}\mathrm{E}[|f(a_1)|^2I_{[|f(a_1)|\le x]}]\ge\frac{1}{n}\}.$$

Assume first that $\mathrm{E}[f^2(a_1)]<\infty$. Thus by Theorem 18.5.2 in [14] and Proposition 2 we have $0<\sigma^2<\infty$ and

$$\mathcal{L}(b_n^{-1}\sum_{k=1}^{n}f(a_k))\to\mathcal{N}(0,1).$$

Now, assume $\mathrm{E}[f^2(a_1)]=\infty$. Then $b_n^2\sim n\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\le b_n]}]$ and $b_n\to\infty$. Let $X_{nk}=f(a_k)I_{[|f(a_k)|\le b_n]}$, $S_n=\sum_{k=1}^n X_{nk}$. By the results in [34] if the following conditions are satisfied

$$\lim_{n\to\infty}n\mathrm{P}[|f(a_n)|>b_n]=0, \tag{3.17}$$

$$\tau_n^2=\mathrm{Var}(\sum_{k=1}^{n}X_{nk})\to\infty, \tag{3.18}$$

$$\lim_{n\to\infty}\tau_n^{-2}\mathrm{E}[\max_{1\le k\le n}(X_{nk}-\mathrm{E}[X_{nk}])^2]=0 \tag{3.19}$$

then

$$\mathcal{L}(\tau_n^{-1}(S_n-n\mathrm{E}[X_{n1}]))\to\mathcal{N}(0,1). \tag{3.20}$$

The condition (3.17) easily follows by the definition of b_n and the slow variation of $\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\le x]}]$ (Cf. Theorem 2, VIII, §9 in [8]). For (3.18) let us observe that by (2.9) and since $\mathrm{E}[f^2(a_1)]=\infty$ thus $\mathrm{E}^2|X_{n1}|=o(\mathrm{E}[X_{n1}^2])$ (Cf. (2.6.14) in [14]) and therefore we have $b_n^2\sim n\mathrm{Var}[X_{n1}]$. Now by

$$\begin{aligned}\tau_n^2&=\mathrm{E}(\sum_{k=1}^{n}(X_{nk}-\mathrm{E}[X_{n1}]))^2\\&=n\mathrm{Var}[X_{n1}]+2\sum_{k=2}^{n}(n-k+1)\mathrm{Cov}[X_{n1}X_{nk}]\\&=n\mathrm{Var}[X_{n1}](1+O(\frac{2}{(1-\varrho)}\frac{n}{b_n^2}\mathrm{E}^2|X_{n1}|))=b_n^2(1+o(1))\end{aligned}$$

(3.18) follows. For the relation (3.19) note that for every $\epsilon\in(0,1)$, some $K>0$ independent of n by the slow variation of $\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\le x]}]$

$$\begin{aligned}&\tau_n^{-2}\mathrm{E}[\max_{1\le k\le n}(X_{nk}-\mathrm{E}[X_{nk}])^2]\\&\quad\le K(b_n^{-2}n\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\le b_n]}I_{[f^2(a_1)I_{[|f(a_1)|\le b_n]}>\epsilon b_n^2]}]+\epsilon)\\&\quad\le K(b_n^{-2}n\mathrm{E}[f^2(a_1)I_{[|f(a_1)|\in(\sqrt{\epsilon}b_n,b_n]}]+\epsilon)\le K(o(1)+\epsilon).\end{aligned}$$

Thus (3.20) holds and since $\mathrm{E}[f(a_1)] = 0$, $\tau_n \sim b_n$ so that $\mathcal{L}(b_n^{-1}S_n) \to \mathcal{N}(0,1)$. If $\mathrm{E}[f(a_1)] \neq 0$ then we consider the sequence $\{f(a_k) - \mathrm{E}[f(a_1)]\}$. It is worth noting that by Theorem 1.9.8 in [3] the sequence $\{b_n\}$ can be replaced by $\{c_n\}$ such that $\lim_n \frac{n}{c_n^2}(c_{n+1}^2 - c_n^2) = 1$.

Conversely, assume $\phi_n(c_n^{-1}\theta)e^{-i\theta\frac{d_n}{c_n}} = \mathrm{E}[e^{i\theta c_n^{-1}\sum_{k=1}^n f(a_k)}]e^{-i\theta\frac{d_n}{c_n}} \to e^{-\frac{\theta^2}{2}}$ thus we have $\widehat{\phi}_n(c_n^{-1}\theta) = \mathrm{E}[e^{i\theta c_n^{-1}\sum_{k=1}^n \widehat{f}(a_k)}] \to e^{-\theta^2}$. By Theorem 3.1 in [18] we have $c_n^2 = nh(n)$, where $h(n)$ is a slowly varying sequence. There is no loss of generality in assuming that $\mathrm{E}[f^2(a_1)] = \infty$ so that $\sup\{x; \mathrm{P}[|\widehat{f}(a_1)| \leq x] < 1\} = \infty$. By Lemma 4 we have $\sup_n c_n^{-q}\mathrm{E}|\sum_{k=1}^n \widehat{f}(a_k)|^q < \infty$, $q \in (0,2)$, so by (2.14) we obtain that the sequence $\{c_n^{-1}\max_{1\leq k\leq n}|\widehat{f}(a_k)|\}$ is uniformly integrable. On the other hand from Lemma 3 we get that

$$\begin{aligned}
&(1-\widehat{\varphi}_1)\widehat{\psi}_1' \lim_n (1 - e^{-n\mathrm{P}[\widehat{f}(a_1) > \epsilon c_n]}) \\
&\quad \leq 4m \lim_n \mathrm{P}[|\widehat{S}_{\lfloor n/m\rfloor+1}| > \frac{\epsilon}{2}c_n] = \frac{4m}{\sqrt{\pi}}\int_{\frac{\epsilon}{2}\sqrt{m}}^{\infty} e^{-u^2/4}du
\end{aligned}$$

and letting $m \to \infty$ it yields $\{c_n^{-1}\max_{1\leq k\leq n}|\widehat{f}(a_k)|\} \to 0$ in probability since the integral tends to 0 faster than exponentially. Moreover, by Theorem 3.5 in [2] the latter convergence takes place in L^1, too. Now, by Proposition 3 (with $q = 1, x = \epsilon, X_k = c_n^{-1}\widehat{f}(a_k)$) we obtain

$$\frac{n}{c_n}\mathrm{E}[|\widehat{f}(a_1)|I_{[|\widehat{f}(a_1)|>\epsilon c_n]}] \to 0.$$

Hence using the notation from the proof of Theorem 1 we get

$$\begin{aligned}
&\sup_n c_n^{-1}\mathrm{E}|Z_n(\epsilon)| \\
&\quad \leq \sup_n c_n^{-1}\mathrm{E}|\sum_{k=1}^n \widehat{f}(a_k)| + \sup_n c_n^{-1}\mathrm{E}|\sum_{k=1}^n \widehat{f}(a_k)I_{[|\widehat{f}(a_k)|>\epsilon c_n]}| \\
&\quad \leq \sup_n c_n^{-1}\mathrm{E}|\sum_{k=1}^n \widehat{f}(a_k)| + \sup_n \frac{n}{c_n}\mathrm{E}[|\widehat{f}(a_1)|I_{[|\widehat{f}(a_1)|>\epsilon c_n]}] < \infty.
\end{aligned}$$

Consequently $\sup_n c_n^{-1}\mathrm{E}|\widehat{Z}_n(\epsilon)| < \infty$ and Lemma 5 (with $p = 2, q = 1$) gives $\sup_n c_n^{-2}\mathrm{E}[\widehat{Z}_n^2(\epsilon)] < \infty$. In view of this it follows from Proposition 2 that $\sup_n \frac{n}{c_n^2}\mathrm{E}[Y_{n1}^2(\epsilon)] < \infty$. Therefore there have to be $n^{-1}c_n^2 \to \infty$ since we assumed $\mathrm{E}[f^2(a_1)] = \infty$.

By Lemma 1 in [11] for $\theta \in \mathbb{R}$ and any integers $m > 0$, $p > 1$ such that

$$\mathrm{E}|e^{i\theta c_n^{-1}\widehat{f}(a_1)} - 1| \leq \min\{\frac{1}{2(1+\widehat{\psi}_1)^2(2m+1)^2}, \frac{1}{2(1+\widehat{\psi}_1)(2pm+1)}\} \tag{3.21}$$

and

$$n\mathrm{E}^2|e^{i\theta c_n^{-1}\widehat{f}(a_1)} - 1| \leq \frac{1}{2}(9 + \sum_{\nu=1}^m \widehat{\psi}_\nu)^{-1}n(1 - \widehat{\phi}_1(c_n^{-1}\theta)) \tag{3.22}$$

we have

$$
\begin{aligned}
&|\widehat{\phi}_n(c_n^{-1}\theta) - \exp\{n(\widehat{\phi}_1(c_n^{-1}\theta) - 1)\}| \qquad (3.23)\\
&\quad \le (9 + \sum_{\nu=1}^{m} \widehat{\psi}_\nu) n\mathrm{E}^2 |e^{i\theta c_n^{-1}\widehat{f}(a_1)} - 1| \exp\{-\frac{1}{2} n(1 - \widehat{\phi}_1(c_n^{-1}\theta))\}\\
&\qquad +(2^{-p} + (6 + \widehat{\psi}_1)\widehat{\psi}_{m+1}) n\mathrm{E}|e^{i\theta c_n^{-1}\widehat{f}(a_1)} - 1|.
\end{aligned}
$$

Now, observe that

$$\sqrt{n}\mathrm{E}|e^{i\theta c_n^{-1}\widehat{f}(a_1)} - 1| \le \sqrt{\frac{n}{c_n^2}}|\theta|\mathrm{E}|\widehat{f}(a_1)| = o(1). \qquad (3.24)$$

By (3.24) we can put $m = p \equiv \lfloor \sqrt[4]{n} \rfloor$ in (3.21) and (3.22) so that by (3.23) we get

$$\overline{\lim}_n \, |\widehat{\phi}_n(c_n^{-1}\theta) - \exp\{n(\widehat{\phi}_1(c_n^{-1}\theta) - 1)\}| = 0.$$

Since $\widehat{\phi}_n(c_n^{-1}\theta) \to e^{-\theta^2}$ thus $n(\widehat{\phi}_1(c_n^{-1}\theta) - 1) \to -\theta^2$. Whence by the proof of Theorem 8.3.1 in [3] we have that $\mathrm{E}[\widehat{f}^2(a_1) I_{[|\widehat{f}(a_1)| \le x]}]$ is a slowly varying function in the sense of Karamata and by Lemma 6 we get that $\mathrm{E}[f^2(a_1) I_{[|f(a_1)| \le x]}]$ varies slowly, too. Further, by the direct part of this proof we know that one can choose $d_n = n\mathrm{E}[f(a_1)]$, which is finite under the slow variation condition. This completes the proof of Theorem 3.

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References

[1] Araujo, A., Giné, E., 1980. The Central Limit Theorem for real and Banach valued random variables, John Wiley & Sons, New York.

[2] Billingsley, P., 1999. Convergence of Probability Measures, 2nd ed., Wiley, New York.

[3] Bingham, N. H., Goldie, C. M., Teugels, J. L., 1987. Regular variation. Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge Univ. Press, Cambridge, UK.

[4] Bradley, R. C., 1988. A central limit theorem for stationary ρ–mixing sequences with infinite variance, Ann. Probab. 16 1, 313–332.

[5] Bradley, R. C., 2005. Basic properties of strong mixing conditions. A survey and some open questions, Probability Surveys 2, 107-144.

[6] Feller, W., 1936. Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung I, Math. Z. 40, 521–559; II, ibidem 42, 301–312.

[7] Feller, W., 1945. Note on the law of large numbers and "fair" games, Ann. Math. Stat. 16 3, 301–304.

[8] Feller, W., 1971. An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed., Wiley, New York.

[9] Gniedenko, B. W., Kołmogorow, A. N., 1957. Rozkłady graniczne sum zmiennych losowych niezależnych, Państwowe Wydawnictwo Naukowe, Warszawa (revised: Russian edition 1949; Hungarian edition 1951).

[10] Gut, A., 2004. An extension of the Kolmogorov-Feller law of large numbers with an application to the St. Petersburg paradox, J. Theor. Probab. 17 3, 769–779.

[11] Heinrich, L., 1987. Rates of convergence in stable limit theorems for sums of exponentially ψ–mixing random variables with an application to metric theory of continued fractions, Math. Nachr. 131, 149–165.

[12] Hitczenko, P., Kwapień, S., Li, W.V., Schechtman, G., Schlumprecht, T., Zinn, J., 1998. Hypercontractivity and comparison of moments of iterated maxima and minima of independent random variables, EJP 3 2, 1-26.

[13] Hoffmann–Jørgensen, J., 1974. Sums of independent Banach space valued random variables, Studia Math. 52, 159–186.

[14] Ibragimov, I. A., Linnik, Yu. V., 1971. Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Gröningen (translated 1st Russian edition 1965).

[15] Iosifescu, M., 1989. On mixing coeficients for the continued fraction expansion, Stud. Cerc. Mat. 41 6, 491–499.

[16] Iosifescu, M., 1990. A survey of the metric theory of continued fractions, fifty years after Doeblin's 1940 paper, Probability theory and mathematical statistics, Vol. I (Vilnius, 1989), Mokslas, 550–572.

[17] Iosifescu, M., Grigorescu, S., 1990. Dependence with complete connections and its applications, Cambridge University Press, Cambridge, UK.

[18] Jakubowski, A., 1993. Minimal conditions in p–stable limit theorems, Stochastic Process. Appl. 44, 291–327.

[19] Khinchine, A. Ya., 1935. Sul domino di attrazione della legge di Gauss, Giorn. Ist. Ital. Attuari 6, 371–393.

[20] Khinchine, A. Ya., 1964. Continued fractions, 3rd ed., University of Chicago Press, Chicago (translated 3rd Russian edition 1961).

[21] Kolmogorov, A. N., 1928. Über die Summen durch den Zufall bestimmter unabhängiger Größen, Math. Ann. 99, 309–319.

[22] Kolmogorov, A. N., 1930. Bemerkungen zu meinen Arbeit "Über die Summen zufälliger Größen", Math. Ann. 102, 484–488.

[23] Kolmogorov, A. N., 1933. Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer-Verlag, Berlin, New York (reprinted: Foundations of the Theory of Probability, 2nd ed., Chelsea Publ. Comp, New York, 1956; 2nd Russian edition: Fundamental concepts of probability theory, Nauka, Moscow, 1974).

[24] Lévy, P., 1935. Propriétés asymptotiques des sommes de variables aléatoires indépendant ou enchainées, J. Math. Pure. Appl. 14, 347–402.

[25] Loève, M., 1960. Probability Theory, 2nd ed., Van Nostrand, New York.

[26] Lin, Z. Y., Lu, C. R., 1996. Limit theory for mixing dependent random variables, Mathematics and its Application v.378, Science Press, Beijing & Kluwer Academic Press, Dodrecht.

[27] Marcinkiewicz, J., Zygmund, A., 1937. Sur les fonctions indépendantes, Fund. Math. 29, 60–90.

[28] Nagaev, S. V., 2000. Probabilistic and moment inequalities for dependent random variables, Teor. Veroyatnost. i Primenen. 45, 194–202.

[29] Peligrad, M., 1990. On Ibragimov–Iosifescu conjecture for φ–mixing sequences, Stochastic Process. Appl. 35, 293–308.

[30] Petrov, V. V., 1996. Limit Theorems of Probability Theory. Sequences of Independent Random Variables, Oxford Studies in Probability 4, Oxford, UK.

[31] Phillip, W., 1988. Limit theorems for sums of partial quoutients of continued fractions, Mh. Math. 105, 195–206.

[32] Samur, J. D., 1989. On some limit theorems for continued fractions, Trans. Amer. Math. Soc. 316, 53–79.

[33] Shao, Q.-M., 1995. Maximal inequalities for partial sums of ρ–mixing sequences, Ann. Probab. 23 2, 948–965.

[34] Szewczak, Z. S., 1992. An invariance principle for uniformly strong mixing stationary sequences when $\mathcal{E}X^2 = \infty$, Probab. Math. Statist. 13, 87-95.

[35] Szewczak, Z. S., 2001. Relative stability for strictly stationary sequences, J. Multivariate Anal. 78 2, 235–251.